Dominating a family of graphs with small connected subgraphs

Yair Caro ∗ Raphael Yuster †

Abstract

Let \( F = \{G_1, \ldots, G_t\} \) be a family of \( n \)-vertex graphs defined on the same vertex-set \( V \), and let \( k \) be a positive integer. A subset of vertices \( D \subset V \) is called an \((F,k)\)-core if for each \( v \in V \) and for each \( i = 1, \ldots, t \), there are at least \( k \) neighbors of \( v \) in \( G_i \) which belong to \( D \). The subset \( D \) is called a connected \((F,k)\)-core, if the subgraph induced by \( D \) in each \( G_i \) is connected. Let \( \delta_i \) be the minimum degree of \( G_i \) and let \( \delta(F) = \min_{i=1}^{t} \delta_i \). Clearly, an \((F,k)\)-core exists if and only if \( \delta(F) \geq k \), and a connected \((F,k)\)-core exists if and only if \( \delta(F) \geq k \) and each \( G_i \) is connected. Let \( c(k,F) \) and \( c_c(k,F) \) be the minimum size of an \((F,k)\)-core and a connected \((F,k)\)-core, respectively. The following asymptotic results are proved for every \( t < \ln \ln \delta \) and \( k < \sqrt{\ln \delta} \):

\[
    c(k,F) \leq n \frac{\ln \delta}{\delta} (1 + o_\delta(1)) \quad c_c(k,F) \leq n \frac{\ln \delta}{\delta} (1 + o_\delta(1)).
\]

The results are asymptotically tight for infinitely many families \( F \). The results unify and extend related results on dominating sets, strong dominating sets and connected dominating sets.

1 Introduction

All graphs considered here are finite, undirected and simple. For standard graph-theoretic terminology the reader is referred to [3]. A major area of research in graph theory is the theory of domination. Recently two books [7, 8] have been published that present most of the known results concerning domination parameters. Among the most popular of these parameters are the “connected domination number”, the “\( k \)-domination number” and the “strong domination number” which are considered in this paper.

A subset \( D \) of vertices in a graph \( G \) is a dominating set if every vertex not in \( D \) has a neighbor in \( D \). \( D \) is called a strong dominating set if every vertex of \( G \) has a neighbor in \( D \). If the subgraph induced by \( D \) is connected, then \( D \) is called a connected dominating set or a connected strong dominating set, appropriately. \( D \) is called a strong \( k \)-dominating set if every vertex of \( G \) has at least \( k \) neighbors in \( D \). The analogous definitions of a \( k \)-dominating set, connected strong

∗Department of Mathematics, University of Haifa-Oranim, Tivon 36006, Israel. email: yairc@macam98.ac.il
†Department of Mathematics, University of Haifa-Oranim, Tivon 36006, Israel. email: raphy@macam98.ac.il
A graph $G$ has a connected dominating set if and only if $G$ is connected; thus $\gamma_c(G)$ is well-defined on the class of connected graphs. The same is true for connected strong domination (assuming the graph has at least two vertices). In order to have a $k$-dominating set, or a strong $k$-dominating set, it is necessary and sufficient that the minimum degree be at least $k$.

The problem of finding small connected dominating sets and small connected strong dominating sets are a major topic of research in the area of graph algorithms, because such sets correspond to the non-leaves of a spanning tree.

There are several results which estimate some of the above-mentioned graph parameters as a function of the minimum degree of the graph. A well-known result of Lovász [9] (see another proof in [2]) states that $\gamma(G) \leq n^{1+\ln(\delta+1)\over \delta+1}$ for every $n$-vertex graph $G$ with minimum degree $\delta > 1$. This result is asymptotically optimal for general graphs $G$. This was shown by Alon [1] who proved by probabilistic methods that when $n$ is large there exists a $\delta$-regular graph with no dominating set of size less than $(1 + o(1))^{1+\ln(\delta+1)\over \delta+1}n$. (We mention here that when $\delta \leq 3$ exact results were obtained in [10, 11].) Caro [4] has considered $k$-domination numbers and showed an analog result to the one obtained by Lovász, under the (obviously necessary) assumption that $\delta >> k$. Thus, he showed that $\gamma(k,G) \leq n^{\ln\delta\over \delta}(1 + o\delta(1))$. Considering connected domination, Caro, West and Yuster [5] have shown by more complicated arguments that the bound obtained by Lovász also holds in this much more restricted case, namely $\gamma_c(k,G) \leq n^{\ln\delta\over \delta}(1 + o\delta(1))$. Their result also supplies a sequential deterministic algorithm which produces a connected dominating set with (at most) this cardinality, in polynomial time. In this paper we present a generalization of all these results which covers, as a special case, all the above-mentioned graph parameters.

Let $F = \{G_1, \ldots, G_t\}$ be a family of graphs which share the same vertex set $V$. A subset of vertices $D \subset V$ is called an $(F,k)$-core if $D$ is a strong $k$-dominating set of each graph in $F$. We call $D$ a connected $(F,k)$-core if $D$ is a connected strong $k$-dominating set of each graph in $F$. Let $c(k,F)$ and $c_c(k,F)$ denote the minimum cardinality of an $(F,k)$-core, and a connected $(F,k)$-core, respectively. Clearly, $c(k,F)$ can be defined if and only if each graph in $F$ has minimum degree at least $k$, and $c_c(k,F)$ can be defined if and only if each graph in $F$ is connected and has minimum degree at least $k$. We prove the following general result:

**Theorem 1.1** Let $k, t$ and $\delta$ be positive integers satisfying $k < \sqrt{\ln \delta}$ and $t < \ln \ln \delta$. Let $F$ be a family of $t$ graphs on the same $n$-vertex set. Assume that every graph in $F$ has minimum degree at least $\delta$. Then:

$$c(k,F) \leq n^{\ln\delta\over \delta}(1 + o\delta(1)).$$
If all graphs in $F$ are connected then:

$$c_c(k, F) \leq n \ln \frac{\delta}{\delta} (1 + o(1)).$$

Note that the lower bound mentioned by Alon shows, in particular, that the bounds obtained in Theorem 1.1 are asymptotically optimal. Moreover, by considering the case $t = 1$ (i.e. $F = \{G\}$) we have that Theorem 1.1 contains, as a special case, all the above-mentioned results. The result of Lovász on $\gamma(G)$ is obtained (in the asymptotic sense) by taking $k = 1$ and using the fact

$$\gamma(G) \leq c_\gamma^*(1, G) = c(1, \{G\}).$$

Caro’s result on $\gamma(k, G)$ is obtained by using the fact

$$\gamma(k, G) \leq c_\gamma^*(k, G) = c(k, \{G\}).$$

The Caro, West and Yuster result on $\gamma_c(G)$ is obtained by taking $k = 1$ and using

$$\gamma_c(G) \leq c_\gamma^*(1, G) = c(1, \{G\}).$$

Our proof of Theorem 1.1 uses a probabilistic approach similar to the proof of the Lovász bound in [2]. However, the proof here is slightly more complicated since we also need to satisfy the connectivity and the commonality requirements. The proof is presented in the next section.

2 Proof of the main result

We begin with a lemma that sharpens a result of Duchet and Meyniel [6], who proved that

$$\gamma(G) \leq \gamma_c(G) \leq 3 \gamma(G) - 2.$$ 

**Lemma 2.1** Let $G$ be a connected graph. If $X$ is a strong $k$-dominating set of $G$ that induces a subgraph with $s$ components, then there exists a connected strong $k$-dominating set of $G$, containing $X$, whose cardinality is at most $|X| + 2s - 2$. In particular,

$$\gamma^*(k, G) \leq \gamma_c^*(k, G) \leq 3 \gamma^*(k, G) - 2.$$

**Proof:** It suffices to show that whenever $s > 1$, we can find at most two vertices in $V \setminus X$ such that adding them to $X$ decreases the number of components by at least one. Partition $X$ into parts $X_1$ and $X_2$ such that $X_1$ and $X_2$ have no edge connecting them. Let $x_1 \in X_1$ and $x_2 \in X_2$ be two vertices whose distance in $G$ is the smallest possible. The distance between $x_1$ and $x_2$ is at most 3, because otherwise, there is a vertex (in the middle of a shortest path from $x_1$ to $x_2$) that has distance at least 2 from both $X_1$ and $X_2$ and has no neighbor in $X$, contradicting the fact that $X$ is, in particular, a dominating set. $\square$

**Proof of Theorem 1.1:** We shall prove the (obviously more difficult) connected $(F, k)$-core version of the theorem, for $t = \lfloor \ln \ln \delta \rfloor$ and $k = \lfloor \sqrt{\ln \delta} \rfloor$. Fix $0 < \epsilon < 1/2$. We shall prove that, for sufficiently large $\delta$, every $F = \{G_1, \ldots, G_t\}$ (the graphs sharing the same vertex set $V$) has an $(F, k)$-core of size at most $(1 + \epsilon)n \ln \frac{\delta}{\delta}$.

Let $p = (1 + \frac{\epsilon}{2}) \frac{n \ln \delta}{2}$ and let $X$ be a random subset of $V$, where each vertex is chosen independently with probability $p$. Let $Y$ be the set of vertices in $V$ that have fewer than $k$ neighbors in $X$ in one of
the graphs \(G_1, \ldots, G_t\). Note that \(X \cup Y\) is a \(k\)-dominating set for each \(G_i\) (although not necessarily a strong one). So let \(Z\) be a minimal set containing \(k\) neighbors of every vertex \(y \in Y\) in each \(G_i\); thus \(|Z| \leq kt|Y|\). Then \(X \cup Y \cup Z\) is strongly \(k\)-dominating in each \(G_i\). Let \(H_i = G_i[X \cup Y \cup Z]\) (the subgraph of \(G_i\) induced by \(X \cup Y \cup Z\)), and let \(c_i\) denote the number of components of \(H_i\). According to Lemma 2.1, we can add at most \(2c_i - 2\) vertices to \(X \cup Y \cup Z\) and obtain a connected strong \(k\)-dominating set of \(G_i\). It follows that there exists a connected \((F,k)\)-core whose size is less than

\[ w = |X| + |Y| + |Z| + 2 \sum_{i=1}^t c_i. \]

We shall estimate the expectations of the summands. Obviously, \(E[|X|] = pn = (1 + \frac{\epsilon}{2})n \ln \delta/\delta\). By examining any \(\delta\) neighbors of a vertex \(v\) in \(G_i\) we see that the probability that \(v\) is adjacent to fewer than \(k\) vertices of \(X\) in \(G_i\) is at most

\[ \sum_{i=0}^{k-1} \binom{\delta}{i} p^i (1-p)^{\delta-i} < \sum_{i=0}^{k-1} (\delta p)^i e^{-p(\delta-k)} = O\left(k(2 \ln \delta)^k \delta^{-1+\epsilon/2}\right), \]

which is at most \(O\left(\delta^{-1+\epsilon}\right)\), so

\[ E[|Y|] = O\left(nt \delta^{-1+\epsilon}\right) = o\left(\frac{n}{\delta}\right) \]

and since \(|Z| \leq kt|Y|\) we also have \(E(|Y| + |Z|) = o(n/\delta)\). Finally, we estimate \(E[c_i]\). Every vertex of \(X \setminus Y\) has at least \(k\) neighbors in \(X\), and hence belongs to a component of \(H_i\) of order at least \(k+1\), so

\[ c_i \leq \frac{1}{k+1} (|X| + |Y| + |Z|) + |Y| + |Z| \]

and thus

\[ E[c_i] \leq \frac{pn}{k+1} + o\left(\frac{n}{\delta}\right) = o\left(\frac{n \ln \delta}{\delta \ln \ln \delta}\right). \]

We therefore have:

\[ E[2 \sum_{i=1}^t c_i] = o\left(\frac{n \ln \delta}{\delta}\right) \]

and hence, by linearity of expectation, \(E[w] = (1 + \frac{\epsilon}{2} + o(1))n \ln \delta/\delta\), which implies that there is an \((F,k)\)-core of size at most \((1 + \epsilon)n \ln \delta/\delta\) for \(\delta\) sufficiently large. \(\square\)

**Acknowledgment**

The authors wish to thank Teresa W. Haynes for valuable references, and the referee for supplying us with a much shorter proof than the original one.
References


