Decomposing oriented graphs into transitive tournaments

Raphael Yuster [*]
Department of Mathematics
University of Haifa
Haifa 39105, Israel

Abstract

For an oriented graph G with n vertices, let $f(G)$ denote the minimum number of transitive subtournaments that decompose G. We prove several results on $f(G)$. In particular, if G is a tournament then $f(G) < \frac{5}{21}n^2(1+o(1))$ and there are tournaments for which $f(G) > n^2/3000$. For general G we prove that $f(G) \le \lfloor n^2/3 \rfloor$ and this is tight. Some related parameters are also considered.

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1 Introduction

All graphs and digraphs considered here are finite and have no loops or multiple edges. For the standard terminology used the reader is referred to [1]. An *oriented graph* is a digraph without directed cycles of length two (antiparallel edges). In other words, it is an orientation of a simple graph. A *tournament* on n vertices is an orientation of K_n. An oriented graph is called *acyclic* if it has no directed cycles. An acyclic tournament is usually called a *transitive tournament*. We denote by TT_k the unique (up to isomorphism) transitive tournament on k vertices.

A *transitive decomposition* of a digraph G is a set of edge-disjoint transitive subtournaments that occupy all the edges of the graph. Namely, each edge of G belongs to precisely one transitive subtournament in the set. Let $f(G)$ denote the minimum size of a transitive decomposition of G. Since a digraph with $e(G) = m$ edges has a trivial transitive decomposition into m copies of TT_2 we always have $f(G) \le e(G)$. The goal of this paper is to study transitive decompositions and to obtain nontrivial bounds for $f(G)$. We note that this problem is closely related to the problem of Erdős, Goodman and Pósa [3] who asked for the minimum number of cliques that decompose a graph G. They proved that if G has n vertices then $\lfloor n^2/4 \rfloor$ cliques always suffice, and this is tight.

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[*] e-mail: raphy@research.haifa.ac.il

Let $f(n,m)$ denote the maximum possible value of $f(G)$ taken over all oriented graphs with n vertices and m edges. Particularly interesting is the value of $f(n, \binom{n}{2})$, namely, the minimal number of transitive tournaments that are needed in order to decompose an n-vertex tournament, in the worst case. For notational convenience we put $f(n) = f(n, \binom{n}{2})$.

In the next section we consider upper and lower bounds for $f(n)$. Notice that it is not obvious at first sight that $f(n) = \Theta(n^2)$. This is because every n-vertex tournament contains many copies of $TT_{\Theta(\log n)}$ which is easy to prove by induction. However, we prove that there are tournaments in which the large transitive subtournaments cannot be packed so as to cover all but $o(n^2)$ edges. In particular, we prove the following.

Theorem 1.1 $\frac{1}{3000}n^2(1+o(1)) < f(n) < \frac{5}{21}n^2(1+o(1))$.

We note that both the upper and lower bounds can be slightly improved but they are still quite far. It seems very interesting to determine $f(n)$ even asymptotically.

In Section 3 we consider the more general parameter $f(n,m)$. Clearly, $f(n,m) = m$ if and only if there is an oriented graph with n vertices and m edges without a TT_3. It is not difficult to construct such graphs for all $m \le \lfloor n^2/3 \rfloor$ (as shown in the beginning of Section 3). We prove, however, that for larger m, $f(n,m)$ is still bounded by $\lfloor n^2/3 \rfloor$.

Theorem 1.2 *$f(n,m) \le \lfloor n^2/3 \rfloor$. $f(n,m) = m$ for $m \le \lfloor n^2/3 \rfloor$.*

Section 4 contains some concluding remarks and some results on related parameters.

2 Decomposing tournaments into transitive subtournaments

Let $r(k)$ denote the minimum integer that guarantees that every tournament with $r(k)$ vertices has a TT_k. A trivial induction argument gives $r(k) \le 2^{k-1}$. Hence, it follows that in a tournament with n vertices, every vertex appears in many copies of $TT_{\Theta(\log n)}$. Unfortunately, as wee shall see, in some cases it is impossible to pack these large transitive tournaments in order to obtain a transitive decomposition with $o(n^2)$ elements. In fact, Erdős and Moser [4] proved, using the probabilistic method, that $r(k) \ge 2^{0.5k(1+o(1))}$. This already shows that we *cannot* expect $f(n) = o(n^2/\log^2 n)$ even for random tournaments. It is easy to show $f(2) = 2$, $f(3) = 4$ an it is well known that $f(4) = 8$ and $f(5) = 14$ [7]. In fact, it is straightforward to construct the unique tournament $\mathcal{T}$ on seven vertices without a TT_4. We shall need the following lemmas in order to prove the upper and lower bounds of Theorem 1.1. Our first lemma is (a simple application of) the seminal result of Wilson [8] for undirected graphs.

Lemma 2.1 *Let k be a positive integer. Then K_n has $\frac{1}{k(k-1)}n^2(1-o(1))$ edge-disjoint copies of K_k.*

In fact, Wilson's theorem shows that if some obvious divisibility conditions hold then there is a K_k-decomposition of K_n, assuming n is sufficiently large.

Our second lemma establishes $f(n)$ for some small values of n.

Lemma 2.2 *$f(2) = 1$, $f(3) = 3$, $f(4) = 4$, $f(5) = 6$, $f(6) = 8$ and $f(7) = 10$.*

Proof: The values of $f(n)$ for $n \leq 5$ are easy exercises. We shall prove the case $f(7) = 10$. The case $f(6) = 8$ is easier. Let S be a tournament with t vertices. If $S = TT_7$ then $f(S) = 1$. If S contains a TT_6 then $f(S) \leq 7$ since a TT_6 already contains 15 of the 21 edges. If S contains a TT_5 and does not contain a TT_6 then let (x, y) be an edge such that the other five vertices induce a TT_5 (the notation (x, y) corresponds to an edge from x to y). Since there is no TT_6, not all edges between y and the other five vertices emanate from y. Hence there is a TT_3 containing (x, y) which is edge-disjoint from the TT_5. Consequently, $f(S) \leq 10$. If S has a TT_4 and does not have a TT_5 then we may assume, without loss of generality, that a, b, c, d are the vertices of a TT_4 and e, f, g are the other vertices. It is not difficult to verify that there are three edge-disjoint TT_3, each containing precisely two vertices from e, f, g. Thus, S decomposes into a TT_4, three TT_3 and six TT_2. Consequently, $f(S) \leq 10$. If S has no TT_4 then $S = \mathcal{T}$ (recall that $\mathcal{T}$ denotes the unique 7-vertex tournament without a TT_4). It is easy to verify that $\mathcal{T}$ has 6 edge-disjoint TT_3. Hence $f(\mathcal{T}) = 9$. We have shown that $f(7) \leq 10$. The following tournament S has $f(S) = 10$. Let $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{5, 6, 7\}$. Orient all edges from A to B, from B to C and from C to A. The orientation of the edge $\{12\}$ and $\{34\}$ is arbitrary. Orient the edges inside C in a cycle. It is easy to check that S has no TT_5, and any TT_3 must contain two vertices from the same part. It follows that $f(S) = 10$. ■

The next two lemmas are needed for the lower bound in Theorem 1.1.

Lemma 2.3 *There exist tournaments that do not have more than $n^2/14$ edge-disjoint TT_4.*

Proof: As before, let $\mathcal{T}$ denote the unique 7-vertex tournament without a TT_4. Assume the vertices of $\mathcal{T}$ are $1, \ldots, 7$. We blow up each vertex of $\mathcal{T}$ into either $\lceil n/7 \rceil$ or $\lfloor n/7 \rfloor$ vertices, so that the total number of vertices is n. Let V_i denote the set of vertices blown up from i. For $x \in V_i$ and $y \in V_j$ the orientation of the edge xy is the same as the orientation of ij in $\mathcal{T}$. The orientation between two vertices in the same set is arbitrary. Since $\mathcal{T}$ has no TT_4, we have constructed a tournament with n vertices in which every TT_4 must contain an edge connecting two vertices from the same set. As the total number of edges with both endpoints in the same set is at most $n^2/14$ the claim follows. ■

Lemma 2.4 *For all $t \geq 2$, K_t has a packing with edge-disjoint copies of K_4 so that the number of unpacked edges is at most $4t - 7$.*

Proof: It is well known that for $t \equiv 1, 4 \bmod 12$, K_t has a K_4 decomposition (see, e.g., [2]). Suppose t is not of this form. We may add or delete s vertices where $1 \le s \le 4$ so as to obtain a graph whose number of vertices is either 1 or 4 modulo 12. In case $t \equiv 2, 5 \bmod 12$ we delete one vertex and $t - 1 \le 4t - 7$ edges. In case $t \equiv 0, 3 \bmod 12$ we add one vertex and the t added edges are on $t/3$ copies of K_4 containing precisely $t \le 4t - 7$ original edges. In case $t \equiv 6 \bmod 12$ we delete two vertices and $2t - 3 \le 4t - 7$ edges. In case $t \equiv 7 \bmod 12$ we delete three vertices and $3t - 6 \le 4t - 7$ edges. In case $t \equiv 8 \bmod 12$ we delete four vertices and $4t - 10 \le 4t - 7$ edges. In case $t \equiv 10 \bmod 12$ we add three vertices x, y, z. We may assume that some K_4 of the decomposition contains x, y, z, w where w is an original vertex. This K_4 contains no original edges. The other K_4's containing one of x, y, z contain precisely $3t - 3 \le 4t - 7$ original edges. In case $t \equiv 11 \bmod 12$ we add two vertices. The unique K_4 containing the two new vertices contains only one original edge and the other K_4's containing a new vertex contain precisely $2t-4$ original edges. Again $(2t - 4) + 1 = 2t - 3 \le 4t - 7$. In case $t \equiv 9 \bmod 12$ we add one vertex and use the case of $10 \bmod 12$ to obtain a packing with K_4 which has $3(t + 1) - 3 = 3t$ unpacked edges. Since $6t/(t + 1) > 5$ there is some vertex which is incident with at least 6 unpacked edges. Deleting this vertex we obtain a graph with t vertices and the number of unpacked edges is now at most $(3t - 6) + (t - 6) = 4t - 12 \le 4t - 7$. ■

Proof of the upper bound in Theorem 1.1 Let T be a tournament with n vertices. By Lemma 2.1 we can pack K_n with $\frac{1}{k(k-1)}n^2(1 - o(1))$ edge-disjoint K_k. Particularly, we can pack T with $\frac{1}{42}n^2(1-o(1))$ edge-disjoint subtournaments each having 7 vertices. By Lemma 2.2, $f(7) = 10$. Thus, each of these subtournaments can be decomposed into at most 10 transitive tournaments. It follows that $f(T) \le \frac{10}{42}n^2(1 + o(1))$.

We note that in [9] it is proved that a tournament on n vertices has at least $0.13n^2(1 + o(1))$ edge-disjoint TT_3. These TT_3 cover $0.39n^2(1+o(1))$ edges which implies an upper bound of 0.24 in Theorem 1.1. This is only slightly inferior to our 5/21 upper bound. By computing specific values of $f(k)$ for larger k one may be able to obtain an improve upper bound, but this approach must converge as suggested by the lower bound.

Proof of the lower bound in Theorem 1.1 Let T be the n-vertex tournament constructed in Lemma 2.3, and recall that T has at most $n^2/14$ edge-disjoint TT_4. Consider a transitive decomposition of T with $k = f(T)$ elements whose vertex sizes are $p_1, \ldots, p_k$. Thus, $\binom{n}{2} = \sum_{i=1}^{k} \binom{p_i}{2}$. By Lemma 2.4, the element whose size is p_i contains a set of edge-disjoint TT_4 covering all but at most $4p_i - 7$ edges. It follows that T has at least

$$\frac{\binom{n}{2} - \sum_{i=1}^{k}(4p_i - 7)}{6}$$

edge-disjoint copies of TT_4. Clearly the last sum is minimized when $\sum_{i=1}^{k} p_i$ is maximized. This happens when all the p_i are equal and their common value p satisfies $p(p - 1) = n(n - 1)/k$. For convenience, put $k = \alpha n(n-1)$. Thus, $p = 1/2 + \sqrt{1/4 + 1/\alpha}$ and the number of edge-disjoint copies of TT_4 is at least

$$n(n-1)\left(\frac{1}{12} + \frac{7}{6}\alpha - \frac{2}{3}\alpha(1/2 + \sqrt{1/4 + 1/\alpha})\right).$$

Taking $\alpha = 1/3000$ gives, for n sufficiently large, more than $0.071438n^2 > \frac{1}{14}n^2$ edge-disjoint TT_4 in T, a contradiction. ■

3 Proof of Theorem 1.2

Consider the Turán graph $T(n,3)$. Recall that this graph is a complete 3-partite graph whose vertex classes are as equal as possible. Hence, it has $\lfloor n^2/3 \rfloor$ edges. Let the vertex classes be V_0, V_1, V_2. We orient all edges from V_i to $V_{(i+1) \bmod 3}$ for $i = 0, 1, 2$. Notice that this orientation does not contain a TT_3. Hence, we have that $f(n,m) = m$ for all $m \le \lfloor n^2/3 \rfloor$.

It remains to show that every oriented graph G with n vertices has $f(G) \le n^2/3$. We prove this by induction on n. The theorem clearly holds for $n = 1$. Assume it holds for oriented graphs with $n-1$ vertices. Let $G = (V,E)$ be a graph with n vertices. For a vertex u, let $d^+(u)$ and $d^-(u)$ be the out and in degrees of u, respectively, and let $d(u) = d^+(u) + d^-(u)$ be the total degree. Let $v \in V$ have minimal total degree. Let G' be the induced subgraph of G on $V - v$. By the induction hypothesis, $f(G') \le \lfloor (n-1)^2/3 \rfloor$. Clearly, $f(G) \le d(v) + f(G')$ since we may trivially decompose the edges incident with v into $d(v)$ copies of TT_2. Thus, if $d(v) \le 2n/3$ we have

$$f(G) \le \lfloor \frac{2}{3}n \rfloor + \lfloor \frac{(n-1)^2}{3} \rfloor = \lfloor \frac{n^2}{3} \rfloor.$$

We may now assume that $d(v) = \lfloor 2n/3 \rfloor + a$ where $a > 0$. It suffices to prove that there are a edge-disjoint copies of TT_3 containing v since this would give $f(G) \le a + (d(v) - 2a) + f(G')$ and we can again use the induction hypothesis to obtain $f(G) \le \lfloor \frac{n^2}{3} \rfloor$.

Without loss of generality, assume $d^+(v) \ge d^-(v)$. Let $N^+(v)$ and $N^-(v)$ be the set of out-neighbors and in-neighbors of v, respectively. Hence, $|N^+(v)| = d^+(v) = \lfloor n/3 \rfloor + b$ where $b > 0$ and $|N^-(v)| = d^-(v) = \lfloor n/3 \rfloor + c$ and note that it may be that $c < 0$. Let H_1 (H_2) be the undirected subgraph of G induced by $N^+(v)$ ($N^-(v)$). The minimum degree of H_1 satisfies

$$\delta(H_1) \ge d(v) - (n - d^+(v)) = \lfloor 2n/3 \rfloor + \lfloor n/3 \rfloor - n + a + b \ge a + b - 1.$$

Similarly,

$$\delta(H_2) \ge d(v) - (n - d^-(v)) = \lfloor 2n/3 \rfloor + \lfloor n/3 \rfloor - n + a + c \ge a + c - 1.$$

We shall use the well known fact that a graph with minimum degree δ has a path of length δ and hence a matching of size $\lceil \delta/2 \rceil$ (see, e.g., [1]). Consider first the case $c < 0$. In this case we must have $b \geq a$ and hence there is a matching of size at least $\lceil (2a-1)/2 \rceil = a$ in H_1. If $c \geq 0$ then there is a matching of size $\lceil (a+b-1)/2 \rceil$ in H_1 and a matching of size $\lceil (a+c-1)/2 \rceil$ in H_2 which together is a matching of size at least a in $N^+(v) \cup N^-(v)$. Now, each element of this matching, together with v, yields a TT_3. We have shown that there are a edge-disjoint copies of TT_3 containing v, as required. ■

4 Concluding remarks

- As mentioned in Section 2, if T is an n-vertex random tournament (the orientation of each edge is chosen uniformly at random, and independently) then $f(T) = \Omega(n^2/\log^2 n)$ almost surely (that is, with probability tending to 1 as n tends to infinity). On the other hand, unlike the general case where Theorem 1.1 shows that $f(n) = \Theta(n^2)$, it is not difficult to show that for every $\epsilon > 0$, if T is an n-vertex random tournament then almost surely $f(T) < \epsilon n^2$ for n sufficiently large. Hence $f(T) = o(n^2)$ almost surely for random tournaments. This follows from the result of [5] which, when applied to our setting, gives that for every fixed positive integer k, there is, almost surely, a packing of T with copies of TT_k so that the number of unpacked edges is only $o(n^2)$. In particular this implies that $f(T) \leq \binom{n}{2}/\binom{k}{2} + o(n^2)$.

- As we define $f(G)$ on general digraphs, we may also define the analog of $f(n,m)$ on this wider class where antiparallel edges are allowed. Notice that in this more general case, there is no interesting analog for $f(n)$ since the complete n-vertex digraph can be trivially decomposed into two edge-disjoint copies of TT_n and hence $f(n, n(n-1)) = 2$ in this case. However, using a similar inductive approach as in the proof of Theorem 1.2 it can be shown that $f(n,m) \leq \lfloor n^2/2 \rfloor$ and $f(n,m) = m$ for all $m \leq \lfloor n^2/2 \rfloor$. The construction here is obtained by the existence of an n-vertex bipartite digraph with m edges for $m \leq \lfloor n^2/2 \rfloor$.

- As mentioned in Section 2, it is possible to slightly improve the upper bound in Theorem 1.1 by computing higher explicit values of $f(k)$. This seems to be a difficult task already for relatively small values of k. However, there is another approach which yields a minor improvement of the upper bound and which requires no additional explicit computations. A *fractional* transitive decomposition of a digraph G is an assignment of nonnegative weights to all the transitive subtournaments of G so that for any edge, the sum of the weights of all the transitive subtournaments that contain the edge is precisely one. The *value* of the fractional transitive decomposition is the sum of all assigned weights. Let $f^*(G)$ be the smallest possible value of a fractional transitive decomposition of G. Trivially, $f^*(G) \leq f(G)$. Let $f^*(n)$ be the fractional analog of $f(n)$. Thus, $f^*(n) \leq f(n)$. Let us first show how $f(k)$ can be used to obtain a nontrivial upper bound for $f^*(k')$, where $k' > k$. Let T be a tournament with k' vertices. There are $\binom{k'}{k}$ subtournaments on k vertices. In each of them we may take a transitive decomposition with at most $f(k)$ elements. Since each edge of T appears in $\binom{k'-2}{k-2}$ subtournament with k vertices, we may assign the value $1/\binom{k'-2}{k-2}$ to each element of each transitive decomposition and obtain a fractional transitive decomposition of T whose value is at most

$$\binom{k'}{k}\frac{1}{\binom{k'-2}{k-2}}f(k)=\frac{k'(k'-1)}{k(k-1)}f(k)\geq f^*(k').$$

However in some cases we can do better. Consider, for example, the case $k=7$ and $k'=64$. By the last inequality we have $f^*(64)\leq 960$. Using the same notation as in Section 2, we have that $r(7)\leq 64$ and, in fact, every vertex of a 64-vertex tournament T is a source or a sink of some TT_7. Hence, T has at least 32 distinct TT_7. Since $f(TT_7)=1$ and $f(7)=10$ we have

$$f^*(64)\leq\frac{1}{\binom{62}{5}}\left(10\left(\binom{64}{7}-32\right)+32\right)=960-\frac{288}{\binom{62}{5}}.$$

Now, using Lemma 2.1 applied to $k=64$ in the proof of Theorem 1.1 would give $f^*(n)\leq(\frac{5}{21}-\frac{1}{14\binom{62}{5}})n^2(1+o(1))$. By Theorem 2 of [6], applied to the family of transitive tournaments the values of $f^*(n)$ and $f(n)$ differ only in $o(n^2)$. Thus,

$$f(n)\leq\left(\frac{5}{21}-\frac{1}{14\binom{62}{5}}\right)n^2(1+o(1)).$$

Although this is only a negligible improvement over the upper bound in Theorem 1.1, the approach presented here may be useful in other settings as well.

- By the same argument as in the last paragraph we have that $f^*(n)/(n(n-1))\leq f^*(n-1)/((n-1)(n-2))$. In particular, this shows that $f^*(n)/n^2$ converges to some limit c. Since $f(n)-f^*(n)=o(n^2)$ we also have that $f(n)/n^2$ converges to c. Theorem 1.1 shows that $\frac{5}{21}>c>\frac{1}{3000}$. We leave as an open problem determining c exactly.

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